Math 666
Bervig/Overgard/Wallin
AS1-js1001.doc

Exercise 1: We need to verify that in the axioms for PG(2,q), the words point and line can be interchanged with no change in meaning after slight wording adjustments.

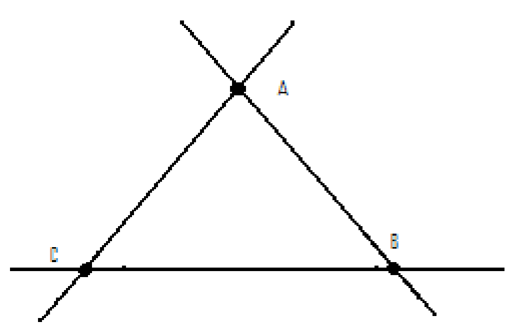
The axioms for PG(2,q) are as follows:
(PG-1) Every line contains exactly $q + 1$ points.
(PG-2) Every point is on exactly $q + 1$ lines.
(PG-3) Any two distinct lines intersect in exactly one point.
(PG-4) Any two distinct points lie on exactly one line.

If we interchange the words point and line, we have the following where PGD stands for PG Dual.
(PGD-1) Every point contains exactly $q + 1$ lines.
(PGD-2) Every line is on exactly $q + 1$ points.
(PGD-3) Any two distinct points intersect in exactly one line.
(PGD-4) Any two distinct lines lie on exactly one line.

These, as we have seen before, need some slight wording adjustments so they fit the English language. They can be reworded as follows.
(PGD-1) Every point is contained by exactly $q + 1$ lines.
(PGD-2) Every line contains exactly $q + 1$ points.
(PGD-3) Any two distinct points are contained in exactly one line.
(PGD-4) Any two distinct lines intersect in exactly one point.

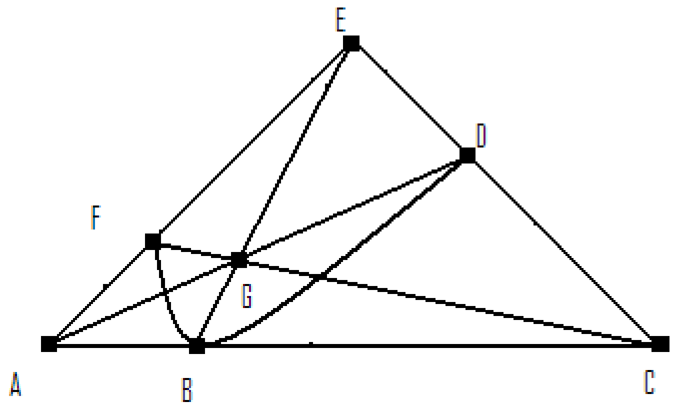
As we can see, (PG-1) is equivalent to (DPG-2), (PG-2) is equivalent to (DPG-1), (PG-3) is equivalent to (DPG-4), and (PG-4) is equivalent to (DPG-3). Thus the axioms are all the same after the words point and line have been interchanged.

Exercise 2: We need to analyze the PG(2,1) geometry. In this geometry each line contains exactly 2 points and each point lies on exactly 2 lines. If we create a model for this, we will see it is like the 3-point geometry. There will be three points and three lines. See the figure below for a graphical representation.



PG(2,1) model

Exercise 3: Here we need to repeat exercise 2 for a PG(2,2) Geometry. In this geometry, every line will contain exactly three points and every point will be on exactly three lines. We will have seven points and seven lines. See below for a graphical representation.



PG(2,2) Model

By axiom PG-1 each line must contain three points, so there must be at least three points, call them A, B, and C. But each of these points must be contained by three lines and these lines must not contain more than one of A, B, or C by PG-4. This makes seven lines necessary. These lines, according to the model are ABC, CDE, EFA, BGE, DGA, FGC, and FBD. All seven of these points must be distinct so that no pair lies on more than one line and we still have three points on each line. We also needed to make sure that each pair of distinct lines intersected in only one point.

Now recall that the Geometry of Fano has the following axioms (taken from the text.)

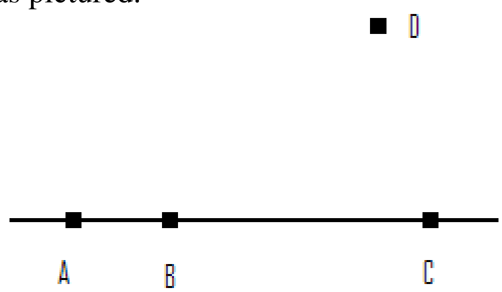
1) There exists at least one line.
2) Every line contains exactly three points.
3) Not all points are on the same line.
4) For each two distinct points, there exists exactly one line containing both of them.
5) Each two lines have at least one point on both of them.

In a table format, we have the following:

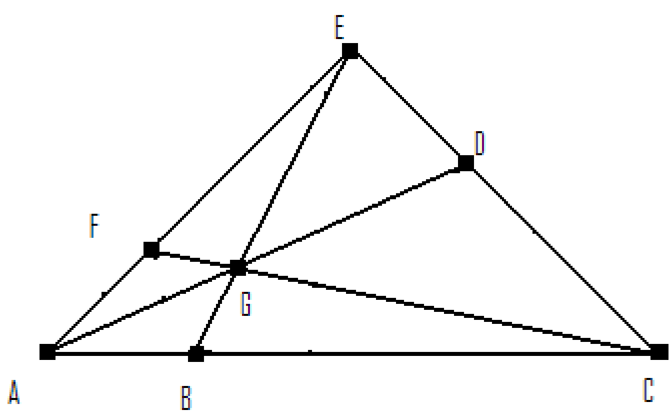
lines (7)	points (7)	point lies on these 3 lines	2 lines intersect 1 point	2 points on each line
123	1	123, 154, 167	123, 154	(1,2), (2,3)
154	2	123, 264, 527	123, 264	(1,5), (5,4)
167	3	123, 365, 374	123, 174	(1,6), (6,7)
264	4	154, 264, 374	154, 374	(2,6), (6,4)
365	5	154, 365, 527	154, 527	(3,6), (6,5)
374	6	167, 264, 365	264, 563	(3,7), (7,4)
527	7	167, 374, 527	167, 473	(5,2), (2,7)

If we look at the model above, we can see that we could create an identical model for the geometry of Fano. We would start by creating one line with three distinct points on it as pictured.

Now since not all points are on the same line, we have to add a fourth point not on this line as pictured.

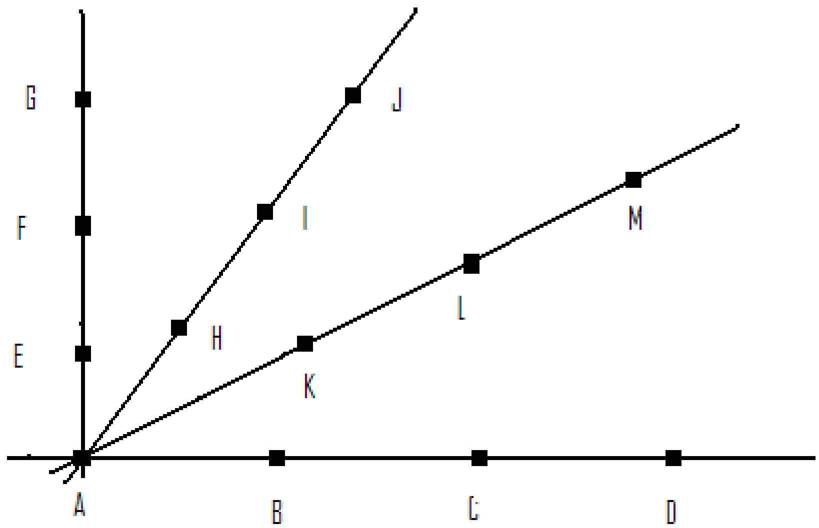


Now we must add three more lines that contain pairs of points AD, BD, and CD. But each of these lines must contain three points so we must add three more points, E, F, and G. Then we will have the following.

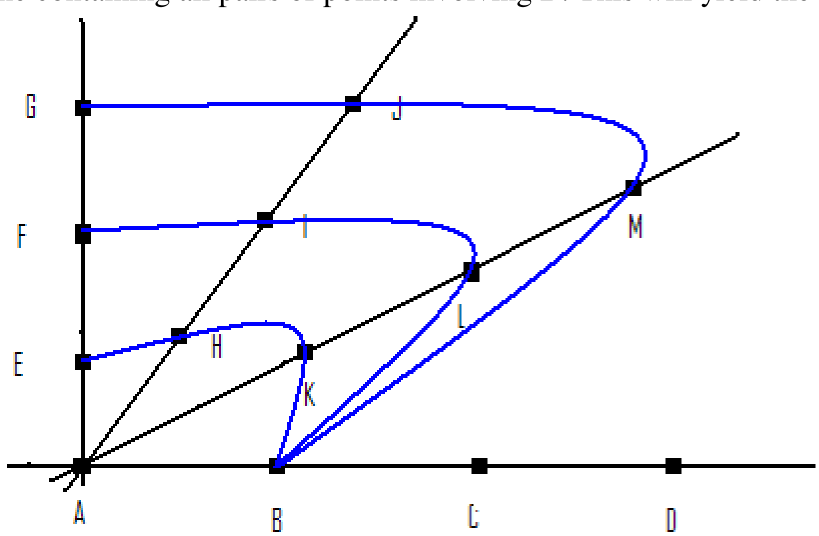


Now, the only problem is that there is not a distinct line containing F and B, or F and D, or B and D. So if we add one more line, FBD, we will have a model that is identical to PG(2,2).

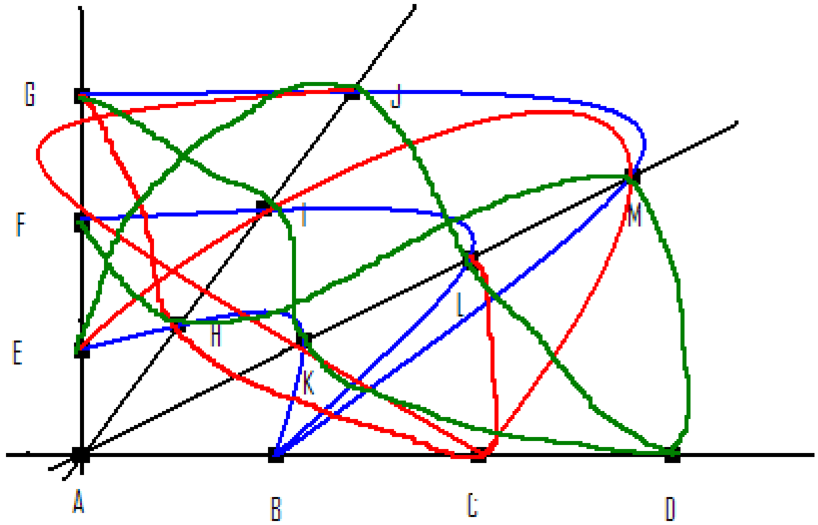
Exercise 4: Here we need to construct PG(2,3). Every line must contain 4 points and every point must lie on 4 lines. If we do this using the brute force method and creating one line with four points, then we can add points as necessary and create the number of lines necessary so that each pair of distinct points lies on exactly one line. We will start with line ABCD, and make sure that point A lies on 4 lines, each having 4 points on them. This will give us the following.



Now we must make sure we do not add any more points or else we will have to create a distinct line containing point A and any other point we add and then A will be contained in more than 4 lines. What we need to do now is create lines containing all other distinct pairs of points. We will start by making sure that B is contained in 4 lines and there is a line containing all pairs of points involving B. This will yield the following.

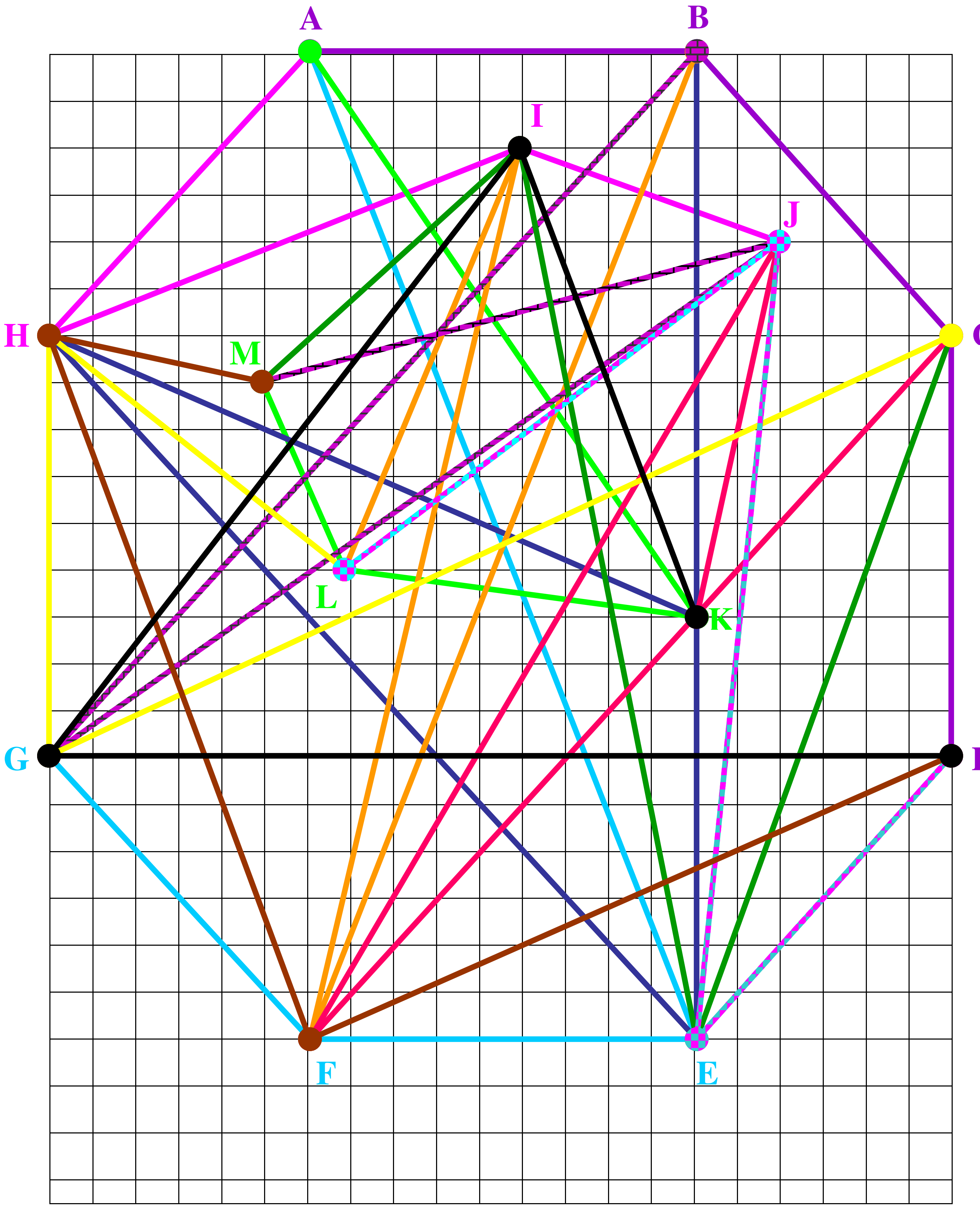


If we do the same for points C and D, we will have the following model.



Here, we have every line containing 4 points, and every point on 4 lines. We also have every a distinct line for every pair of points. The points are: A, B, C, D, E, F, G, H, I, J, K, L, and M. The lines are ABCD, AEFG, AHIJ, AKLM, BEHK, BFIL, BGJM, CEIM, CFJK, CGHL, DEJL, DFHM, and DGIK.

An alternate model for number 4 follows on the next page:



ABCD, AEFG, AHIJ, AKLM, BEHK, BFIL, BGJM, CEIM, CFJK, CGHL, DEJL, DFHM, & DGIK.

Exercise 5: Possible Conjecture: For PG(2,q), q = 4, 5, . . . these geometries will exist and will be self dual and will have $q(q+1)+1$ points and $q(q+1)+1$ lines.
I could possible decide this by trying it for several values of q but this would only be a convincing demonstration, not an actual proof. I could possibly try to do an inductive proof, although I'm not sure how this would work in this situation.

- OR -

I conjecture that PG(2,q) for q = 4,5,… exists since by the leading sentence "Our basic model for a self-dual projective geometry is denoted by PG(2,q) where *q is a positive integer…*" and q = 4,5,… are positive integers. There were 3 points & lines in PG(2,1), 7 points & lines in PG(2,2), and 13 points & lines in PG(2,3), so by that pattern there would be 21 points & lines in PG(2,4) and 31 points & lines in PG(2,5). I might decide by assuming they don't exist and look for a contradiction.

Exercise 6: $N(q) = q^2 + q + 1$

$N(q) = q^2 + q + 1$

From exercise 2: q = 1
$N(1) = 1^2 + 1 + 1$
$= 3$ and yes there are 3 points

From exercise 3: q = 2
$N(2) = 2^2 + 2 + 1$
$= 7$ and yes there are 7 points

From exercise 4: q = 3
$N(3) = 3^2 + 3 + 1$
$= 13$ and yes there are 13 points

Exercise 7: We need to fill in the details of the following proof.

Proof (Number of Points): Let O (for origin) be any point of PG(2,q).
7a) Let $V_0, V_1, \ldots, V_q$ be the lines containing O (these will be called vertical lines.) We know there must be q + 1 lines containing O by axiom PG-2.
7b) Now, by axiom PG-1, each line must contain q + 1 points, so each of these vertical lines must contain exactly q points other than O. Let $O, P_{(1,j)}, P_{(2,j)}, \ldots, P_{(q,j)}$ be the points on line V_j for $j = 0,1,\ldots q$.
7c) Now we will show that the points $P_{(i,j)}$ for i = 1, . . ., q and j = 1, . . . , q are all distinct. First consider the points $P_{(i,j)}$ and $P_{(k,j)}$ where $i \neq k$. These two points lie on the same vertical line. They must be distinct or there will not be q + 1 points on that line. Therefore, all points on the same vertical line must be distinct. Now consider two points

on different vertical lines, $P_{(i,j)}$ and $P_{(k,l)}$ where $j \neq l$. Assume these points are not distinct, but that $P_{(i,j)} = P_{(k,l)}$. Now if these points are not distinct but are the same point, then because this point lies on two different vertical lines, then point *O* will be paired with this point on more than one line. This is a violation of axiom PG-4. Therefore, these points must be distinct. Thus we have that any two points on the same vertical line must be distinct, and any two points on different vertical lines must be distinct, so the points $P_{(i,j)}$ for *i* = 1, . . ., *q* and *j* = 1, . . . , *q* are all distinct.

7d) Now we know that there are at least 1 + *q*(*q* + 1) points for PG(2,q). Suppose there is another point *R* that is distinct from *O* and $P_{(i,j)}$ for *i* = 1, . . ., *q* and *j* = 1, . . . , *q*. Then there must be a line containing the pair of points *O* and *R*. But this cannot be any of the lines we have already defined since they all already contain four points each. If this point *R* were on one of those lines, it would violate axiom PG-1. This means there must be a distinct line containing the pair of points *O* and *R*. But if this is the case then *O* is contained in *q* + 2 lines which contradicts axiom PG-2. Therefore, we cannot have point *R* in this geometry and so PG(2,q) can have at most 1 + *q*(*q* + 1) points. This means PG(2,q) can have at most and must have at least 1 + *q*(*q* + 1) points so it must have exactly 1 + *q*(*q* + 1) points. Therefore $N(q) = q^2 + q + 1$.

Exercise 8: Prove Proposition (Vertical Lines).

(1) A line is vertical iff it contains (0,0) and the other q points have the same second coordinate.
(2) A line is vertical iff it contains (0,0) and exactly one column of the array.

Proof: (1) and (2): By Definition in Projective Geometry (Pitts, Yasskin, p.3): "Fix a point O in PG(2,q) and call it the origin. Any line which contains O is called vertical". Since both (1) and (2) contain (0,0) and other points, they are vertical.

If the other q points have the same second coordinate, q=1,…,q, and it also contains (0,0), then again by the above definition it is vertical. If exactly one column of the array and (0,0) are on a line, then the line is vertical because then they also, by definition of a column, have the same second coordinate. The array is displayed by using (0,0), and then also P(i,j) where i is the row number and j is the column number.

Exercise 9: Prove the following Proposition:

(1) No horizontal line contains the point (0,0).
(2) A line is horizontal iff it contains exactly one point on each vertical line.
(3) A line is horizontal iff it contains exactly one point with second coordinate *j* for each *j* = 0, 1, . . ., *q*.
(4) A line is horizontal iff it contains exactly one point in each column of the array,
(5) For each pair of points (m,0) in column 0 (where $m \neq 0$) and (n,1) in column 1, there is a unique line denoted H(m,n) containing (m,0) and (n,1).

Proof:

(1) Any line containing *O* is vertical, so if a line contains *O* it will not be horizontal. Therefore no horizontal line contains *O*.

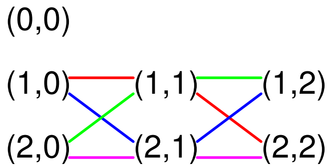
(2) First prove if a line is horizontal then it contains exactly one point on each vertical line. Note that there are q+1 vertical lines. We also know that each line contains q+1 points. Suppose a line H is horizontal and does not contain exactly one point from each vertical line, but contains two points from one vertical line and no points from another vertical line. Then the pair of points from the same vertical line will both be contained in two distinct lines, one vertical and one horizontal. This contradicts axiom PG-4 and can therefore not happen. Thus each horizontal line must contain exactly one point from each vertical line.
Next we prove if a line contains exactly one point on each vertical line then it is horizontal. Suppose some line H contains exactly one point on each vertical line, and assume H is vertical. If it contains one point on each vertical line, then the points on H will not all have the same second coordinate, which contradicts the proposition from exercise 8. Thus H cannot be vertical and must be horizontal.

(3) First prove if a line is horizontal, it contains exactly one point with second coordinate j for each $j = 0, 1, \ldots, q$. Suppose some line H is horizontal. Then we know H contains exactly one point from each vertical line by (2) above. Now the vertical lines are represented by the second coordinate of each point in the array, so if H contains exactly one point from each vertical line, then each point will have a different second coordinate. Also, since H contains exactly one point from EACH vertical line, then each of the coordinates $j = 1, \ldots, q$ must be associated with some point on H. Thus if H is a horizontal line, then it contains exactly one point with second coordinate j for each $j = 0, 1, \ldots, q$.
Next prove if a line contains exactly one point with second coordinate j for each $j = 0, 1, \ldots, q$ then it is a horizontal line. Suppose some line H contains exactly one point with second coordinate j for each $j = 0, 1, \ldots, q$. Then all of the points on H will have a different second coordinate. By the proposition in exercise 7, part (1), H must not be vertical and must be horizontal. Thus if a line contains exactly one point with second coordinate j for each $j = 0, 1, \ldots, q$ then it is a horizontal line.

(4) First prove if a line is horizontal, it contains exactly one point in each column of the array. Consider some horizontal line H. By (3) above, we know H contains exactly one point with second coordinate j for each $j = 0, 1, \ldots, q$. Now the second coordinate represents the vertical line the point is on, and vertical lines are represented by the columns of the array. Thus H must contain exactly one point in each column of the array.
Next prove if a line contains exactly one point in each column of the array, then it is horizontal. Suppose some line H contains exactly one point in each column of the array. Then H contains exactly one point in each vertical line since the columns represent the vertical lines. Then by (2) above, H must be horizontal.

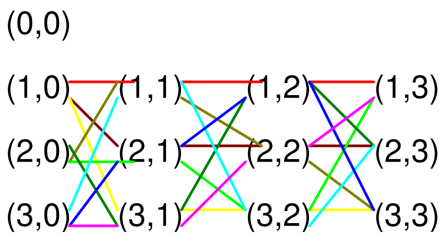
(5) Prove for each pair of points (m,0) in column 0 (where $m \neq 0$) and (n,1) in column 1, there is a unique line denoted H(m,n) containing (m,0) and (n,1). Consider the pair of points (m,0) in column 0 (where $m \neq 0$) and (n,1) in column 1. Since $m \neq 0$ then the first point is not O, and since the second coordinate on the second point is 1, it is also not O. Also, since these points have different second coordinates, they lie on different vertical lines and are distinct points. Thus by axiom PG-4, there is a unique line containing the points (m,0) in column 0 (where $m \neq 0$) and (n,1) in column 1.

Exercise 10: Draw all 4 horizontal lines of PG(2,2) in its array.



Note that this diagram shows only the horizontal lines, not the vertical lines.

Exercise 11: Draw all 9 horizontal lines of PG(2,3) in its array.



Again note that only the horizontal lines are shown. The lines are color-coded.

Exercise 12:

(1) Assume H^k(m,n) = H^k(m,n').. then (m,n) = (m,n') and they would be in the same column so (m,n) will be on a horizontal line with (m, n+1) and (m,n') will be on a horizontal line with (m, n+1) too. This can't happen because there is only one horizontal line with (m,n) and (m, n+1) so the horizontal line with the two points (m,n) and (m, n+1) is the same horizontal line with the two points (m,n') and (m.n+1) and thus the numbers H^k(m,n) for n= 1,2,…,q are all distinct.

(2) Assume H^k(m,n) = H^k(m',n) so m = m' and both (m,n) and (m',n) lie in the nth column. (m,n) lies n the horizontal line with (m, n+1) and so does (m',n) lie on the horizontal line with (m, n+1). There points (m',n) and (m,n) must be distinct and the number H^k(m,n) for m = 1,2,…,q are all distinct.

(3) Suppose (H^k(m,n), H^l(m,n)) = (H^k(m',n'), H^l(m',n')) with k $\neq$ l, then H^k(m,n) = H^l(m',n') which means each set lies in the same column and row. To be a horizontal line, n $\neq$ n' but (H^k(m,n), H^l(m,n)) lies on the horizontal line with (H^k(m+1,n+1) = H^l(m+1,n+1)) and likewise (H^k(m',n'), H^l(m',n')) lies on the horizontal line (H^k(m+1,n+1), H^l(m+1,n+1)) so the q^2 ordered pairs (H^k(m,n), H^l(m,n)) for m,n = 1,…,q are all distinct.

Exercise 13: Note: there are no repetitions in any row or column

$$\begin{bmatrix} 1 & 2 & \cdots & q \\ 2 & 3 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ q & 1 & \cdots & q-1 \end{bmatrix}$$

Exercise 14: Show that the two 2 X 2 matrices are not orthogonal.

$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ and $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$ The matrix G = $\begin{vmatrix} (1,2) & (2,1) \\ (2,1) & (1,2) \end{vmatrix}$. Since this has repeated entries, the two original Latin squares of order 2 are not orthogonal.

Exercise 15: Show that the two 3 X 3 matrices are orthogonal.

$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$ The matrix G = $\begin{vmatrix} (1,1) & (3,2) & (2,3) \\ (2,2) & (1,3) & (3,1) \\ (3,3) & (2,1) & (1,2) \end{vmatrix}$. Since this has NO repeated entries, the two original Latin squares of order 3 are orthogonal.

Exercise 16: Identify all complete sets of Latin squares for the case q = 2.

There are two complete sets: $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$, and $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$.

Exercise 17: Complete the construction of PG(2,2) where A_1 is the first matrix in the equation.

$A_1 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ $A_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

First, build an array and label the points accordingly:

(0,0)

(1,0) (1,1) (1,2)

(2,0) (2,1) (2,2)

The vertical lines are VO: {(0,0), (1,0), (2,0)}
V1: {(0,0), (1,1), (2,1)}
V2: {(0,0), (1,2), (2,2)}

The horizontal lines are H(1,1): {(1,0), (1,1), (1,2)}
H(1,2): {(1,0), (2,1), (2,2)}
H(2,1): {(2,0), (1,1), (2,2)}
H(2,2): {(2,0), (2,1), (1,2)}

Exercise 18: Use A_1 as the second matrix.

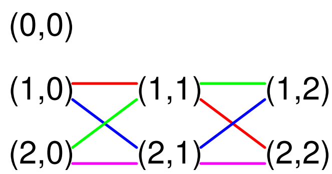
$$A_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \qquad A_1 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

The vertical lines are VO: {(0,0), (1,0), (2,0)}
V1: {(0,0), (1,1), (2,1)}
V2: {(0,0), (1,2), (2,2)}

The horizontal lines are H(1,1): {(1,0), (2,1), (1,2)}
H(1,2): {(1,0), (1,1), (2,2)}
H(2,1): {(2,0), (1,1), (1,2)}
H(2,2): {(2,0), (2,1), (2,2)}

What changed? The horizontal lines are different when one switches the matrices.

In comparison to exercises 3 and 10:



The first model would have had horizontal lines straight across the top (1,0), (1,1), (1,2); and through (2,0), (1,1), (2,2); also through (2,0), (2,1), (1,2); and lastly (1,0), (2,1), (2,2). Number 10's model represents number 18 here.

Exercise 19: Complete the construction of PG(2,3) where

$$A_1 = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} \text{ and } A_2 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}.$$

First, build an array and label the points accordingly:

(0,0)

(1,0) (1,1) (1,2) (1,3)

(2,0) (2,1) (2,2) (2,3)

(3,0) (3,1) (3,2) (3,3)

The vertical lines are VO: {(0,0), (1,0), (2,0), (3,0)}
V1: {(0,0), (1,1), (2,1), (3,1)}
V2: {(0,0), (1,2), (2,2), (3,2)}
V3: {(0,0), (1,3), (2,3), (3,3)}

The horizontal lines are
H(1,1): {(1,0), (1,1), (1,2), (1,3)}
H(1,2): {(1,0), (2,1), (3,2), (2,3)}
H(1,3): {(1,0), (3,1), (2,2), (3,3)}
H(2,1): {(2,0), (1,1), (2,2), (2,3)}
H(2,2): {(2,0), (2,1), (1,2), (3,3)}
H(2,3): {(2,0), (3,1), (3,2), (1,3)}
H(3,1): {(3,0), (1,1), (3,2), (3,3)}
H(3,2): {(3,0), (2,1), (2,2), (1,3)}
H(3,3): {(3,0), (3,1), (1,2), (2,3)}

Now, switching the matrices:

$A_2 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$ and $A_1 = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$ yields the same vertical lines but these horizontal lines:

The horizontal lines are
H(1,1): {(1,0), (1,1), (1,2), (1,3)}
H(1,2): {(1,0), (2,1), (2,2), (3,3)}
H(1,3): {(1,0), (3,1), (3,2), (2,3)}
H(2,1): {(2,0), (1,1), (2,2), (2,3)}
H(2,2): {(2,0), (2,1), (3,2), (1,3)}
H(2,3): {(2,0), (3,1), (1,2), (3,3)}
H(3,1): {(3,0), (1,1), (3,2), (3,3)}
H(3,2): {(3,0), (2,1), (1,2), (2,3)}
H(3,3): {(3,0), (3,1), (2,2), (1,3)}

Exercise 20: Construct PG(2,4).

$$A_1 = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix} \quad A_2 = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{vmatrix} \quad A_3 = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{vmatrix}$$

First, build an array and label the points accordingly:

(0,0)

(1,0) (1,1) (1,2) (1,3) (1,4)

(2,0) (2,1) (2,2) (2,3) (2,4)

(3,0) (3,1) (3,2) (3,3) (3,4)

(4,0) (4,1) (4,2) (4,3) (4,4)

The vertical lines are VO: {(0,0), (1,0), (2,0), (3,0), (4,0)}
V1: {(0,0), (1,1), (2,1), (3,1), (4,1)}
V2: {(0,0), (1,2), (2,2), (3,2), (4,2)}
V3: {(0,0), (1,3), (2,3), (3,3), (4,3)}
V4: {(0,0), (1,4), (2,4), (3,4), (4,4)}

The horizontal lines are H(1,1): {(1,0), (1,1), (1,2), (1,3), (1,4)}
H(1,2): {(1,0), (2,1), (2,2), (2,3), (2,4)}
H(1,3): {(1,0), (3,1), (3,2), (3,3), (3,4)}
H(1,4): {(1,0), (4,1), (4,2), (4,3), (4,4)}
H(2,1): {(2,0), (1,1), (2,2), (3,3), (4,4)}
H(2,2): {(2,0), (2,1), (1,2), (4,3), (3,4)}
H(2,3): {(2,0), (3,1), (4,2), (1,3), (2,4)}
H(2,4): {(2,0), (4,1), (3,2), (2,3), (1,4)}
H(3,1): {(3,0), (1,1), (3,2), (4,3), (2,4)}
H(3,2): {(3,0), (2,1), (4,2), (3,3), (1,4)}
H(3,3): {(3,0), (3,1), (1,2), (2,3), (4,4)}
H(3,4): {(3,0), (4,1), (2,2), (1,3), (3,4)}
H(4,1): {(4,0), (1,1), (4,2), (2,3), (3,4)}
H(4,2): {(4,0), (2,1), (3,2), (1,3), (4,4)}
H(4,3): {(4,0), (3,1), (2,2), (4,3), (1,4)}
H(4,4): {(4,0), (4,1), (1,2), (3,3), (2,4)}

Exercise 21: PG(2,5) exists because I have produced q-1 = 4 pairwise (mutually) orthogonal Latin Squares.

Theorem 1 in "The History of the Problem" says let $S_1, S_2, \ldots, S_t$ be a set of *t* mutually orthogonal Latin Squares of order n ≥ 3, then t ≤ n-1 gives a complete orthogonal set. Here n -1 = 4 = *t*.

Theorem 2 tells us that for $n \geq 3$ we may construct a projective plane of order n if and only if we may construct a complete set of n-1 mutually orthogonal Latin Squares of order n. n= 5 in our case and we found n -1 = 4 mutually orthogonal Latin Squares so we've satisfied both theorems and hence PG(2,5) exists.

Theorem 3: If $n \equiv 1,2 \bmod 4$ then a necessary condition for the existence of a finite projective plane of order n is that integers x,y exist satisfying $n = x^2 + y^2$ .5 $\equiv 1 \bmod 4$ and $1^2 + 2^2 = 5$, so Theorem 3 is satisfied as well.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 2 \\ 2 & 4 & 5 & 1 & 3 \\ 3 & 5 & 1 & 2 & 4 \\ 4 & 1 & 2 & 3 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 & 3 \\ 2 & 5 & 1 & 3 & 4 \\ 3 & 1 & 2 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \\ 4 & 3 & 5 & 1 & 2 \\ 5 & 4 & 1 & 2 & 3 \end{bmatrix}$$

These matrices were constructed with [1 2 3 4 5] being column 1 in each. Then for the first matrix vertical shifts of one place down were used to get each succeeding column. For the second matrix, vertical shifts of 2 places down were used to get each succeeding column, for the third matrix, vertical shifts of 3 places down were used to get each succeeding column, and for the fourth matrix, vertical shifts of 4 places down were used to get each succeeding column. This set of matrices is pairwise orthogonal.

Exercise 22: We know that the PG(2,q) conjecture is true for q = 1, 2, 3, 4, 5, 7, 8, 9, 11. We know it's true for 2, 3, 5, 7 and 11 because they are prime numbers. We know it's true for 4, 8, and 9 because they are positive powers of primes. That is to say $4 = 2^2$; $8 = 2^3$; and $9 = 3^2$.

This conjecture is not true for all integers. It has been proven to be not true for q = 6 and not proven true for q = 10.

Here is what we know about the q = 6 case.

For the conjecture to be true we would need to find 5 pairwise orthogonal matrices. We cannot even find one pair. Using the algorithm to find these matrices, the first two would look like those below:

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 6 & 5 & 4 & 3 & 2 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & 2 & 1 & 6 & 5 & 4 \\ 4 & 3 & 2 & 1 & 6 & 5 \\ 5 & 4 & 3 & 2 & 1 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{vmatrix}$$

Pairwise they make

$$\begin{vmatrix} (1,1) & (2,6) & (3,5) & (4,4) & (5,3) & (6,2) \\ (2,2) & (3,1) & (4,6) & (5,5) & (6,4) & (1,3) \\ (3,3) & (4,2) & (5,1) & (6,6) & (1,5) & (2,4) \\ (4,4) & (5,3) & (6,2) & (1,1) & (2,6) & (3,5) \\ (5,5) & (6,4) & (1,3) & (2,2) & (3,1) & (4,6) \\ (6,6) & (1,5) & (2,4) & (3,3) & (4,2) & (5,1) \end{vmatrix}$$

Since there are duplicate PAIRS, they are not pairwise orthogonal.

At this point we know that $6 \equiv 2 \pmod 4$ and that 6 is not a sum of two integer squares, so by the Bruck-Ryser Theorem, it would seem that there does not exist a projective geometry for q = 6. The Bruck-Ryser Theorem appears to have taken Euler's conjecture a step farther, and it also appears to have been proven even though the proof is not given in the article. We also know that many attempts have been made to construct a complete set of Latin Squares of order 6 by hand and to date none have been successful.

As to the case where q = 10, it does seem to be a rather daunting task to construct, by hand, a complete set of orthogonal Latin Squares of order 10. We know that a pair of them was constructed in 1959 by Parker (this was stated in *The History of the Problem.*) However, creating a complete set is something else altogether. The Bruck-Ryser Theorem does not tell us that we can definitely construct a complete set of orthogonal Latin Squares of order 10, it really only tells us that it may be possible. According to Euler's conjecture, because we know that the conjecture would hold true for any integer q such that $q = x^2 + y^2$ where x and y are integer AND if $q \equiv 1,2 \pmod 4$. Since $10 = 1^2 + 3^2$ we test: $10 \equiv 2 \pmod 4$ true so the conjecture (if true) does not hold for q = 10.

After scanning Dr. Lam's article, *The Search for a Finite Projective Plane of Order 10*, it seems that the main problem is the amount of time needed for even a high-speed computer to attempt this. It also seems (and this is my stance on the matter) that this case has not been settled conclusively. As we speak, technology is being developed and new algorithms are being written. Should a usable and practical quantum computer ever become a reality, then it is certainly possible that the existing algorithm or a new one could be used to create (or at least attempt to create) a complete set of orthogonal Latin Squares of order 10. If this should ever happen, then we might finally know for sure one way or another if this can be done.