Simplifying Rational Expressions
and
Solving Rational Equations

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This paper investigates problems associated with the teaching of rational expressions and equations in a college level mathematics course. Common misconceptions and gaps that mislead the students are addressed, and the basic differences between simplifying an expression and solving an equation are noted and explored. Student activities aimed to help the student learn how to handle rational expressions and equations are presented.

Deficiencies and misconceptions from earlier coursework can have a big effect on the student and therefore should affect how the material is presented. (Allen, 2007; Nite, 2007). Therefore, it is imperative that the instructor address both common misconceptions as well as the content to correct the misconceptions and connect this knowledge to the current material. It is important to "focus on the students-their needs, backgrounds, interests and particularly their existing mathematical understandings." (Beswick, 2007).

This is accomplished by showing the students the similarity that exists between rational expressions and simple numeric fractions, and also between more difficult rational equations and simpler fractional equations. If we as educators can build a bridge between these, the student can connect and handle the more complicated problems.

The key to effective teaching is the adoption of a student-centered Socratic method of teaching. "Inquiry, or investigative methods in mathematics teaching … demand activity, offer challenges to stimulate mathematical thinking and create opportunities for critical reflection on mathematical understanding." (Jaworski, 2006). Therefore, a type of inquiry-based methodology is highly recommended. Research shows that "…as students are solving a problem, they need to implement strategies, use resources, and evaluate their progress so that they are aware of and critically examining their own decision making." (Underwood, 2005). So, if the students know they are expected to participate, they do. Their minds are better focused, and more of the material is learned.

These are some of the problem areas associated with rational expressions and equations:

- Identifying the difference between expressions and equations
- How to find the LCD
- How to use the LCD for adding and subtracting
- How to use the LCD for solving rational equations
- How to factor polynomials in preparation of simplifying or solving
- Canceling out the entire numerator and forgetting to put a "1"
- Forgetting to check for excluded values (i.e.: what values of the variable make the denominator zero)
- Forgetting to place the numerator of a fraction after a subtraction into parentheses, and distributing the minus sign appropriately
- Understanding that $\frac{a-b}{b-a} = \frac{-(b-a)}{b-a} = -1, \quad a \neq b$

Addressing these problems is critical to successfully teaching the material. To help the student, here are some steps to follow:

1. **Determine if it is rational.** (How?). Then determine if it is an expression or an equation.
2. **Expression: We wish to simplify**. If adding or subtracting, find the LCD (How?). Compare to simple fractions. Prime factorization. If multiplying or dividing, we need to prime factor and then cancel. If dividing, we need to multiply by the reciprocal of the second number.
3. **Equation: We wish to solve**. Multiply both sides by the LCD. (Why?)

The areas selected for investigation are listed above. They include how to identify and differentiate rational expressions and rational equations, how to simplify expressions (activity sheets herein are limited to multiplication and division of rational expressions) and how to solve equations.

There are many ways to adequately address the myriad of other problems that college level students may run into, such as ways to work on the problem solving skills and critical thinking skills through the introduction of appropriate word problems. Introduction of a few word problems that require rational equations can be done at the beginning of the lecture and activity time; and the students can be reminded afterward of the practical applications of rational equations. This aspect should be given attention and much of the teaching can be worked around word problems (Potts, 1994), but the student activity sheets submitted with this paper do not involve word problems as the paper can only address a limited number of the potential problem areas.

It is recommended that calculators be used to check simplified expressions. This can be accomplished by evaluating the final answer for a specific value of x, and trying the same value of x in the original unsimplified version to check for equality. Likewise, calculators can be used to check the solution to a rational equation. Calculators used in this way will help. Indeed, "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning… Technology should be used widely and responsibly, with the goal of enriching student's learning of mathematics…. [They] furnish visual images of mathematical ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately… When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving." (NCTM, 2006). As with the concept of using word problems to connect the mathematics to a practical application and work on critical thinking skills, the use of calculators at this level is important. Nonetheless, the activity sheets only briefly mention their use due only to the limited scope of this paper.

Activities, in addition to the lecture and extensive question and answer time, include:

- Word problems worked in small groups to enhance learning
- Vocabulary review (link to earlier algebraic topics such as *rational numbers*)
- Factoring polynomials review (link to FOIL and the distributive property)
- Finding the LCD (link to prime factoring)
- When to use the LCD (link to numeric fractions)
- Simplifying expressions. Break these down to
  - Addition and subtraction (Key – LCD)
  - Multiplication (Key – Prime factorization)
  - Division (Key – Multiply by the reciprocal)
  - Calculator to check through evaluation
- Solving equations.
  - Find the LCD
  - Multiply both sides by the LCD
  - Solve
  - Check using a calculator

With the use of connecting to ideas learned in the past that can be more readily identified with and comprehended, the student is guided into understanding of the new lecture material.

Student Activity #1 – Getting Started

Answer the following questions related to the lecture and discussion. Use complete sentences and give an example.

1. What is a polynomial?

2. What is a monomial?

3. What is a rational number?

4. What is a rational expression?

5. What is a rational equation?

6. What are the steps for factoring polynomials:

7. When adding rational expressions, what must you do?

8. When solving for rational equations, what must you do?

9. a) Explain how to go about finding the LCD of these fractions in words:

$$\frac{5}{162} \text{ and } \frac{2}{21}$$

b) Now, find the LCD of $\frac{5}{162} \text{ and } \frac{2}{21}$:

c) Now, add: $\frac{5}{162}+\frac{2}{21}$

10. a) Explain how to find the LCD of these rational expressions in words:

$$\frac{x^2}{2a-6}, \frac{x}{a^2-a-6}, \text{ and } \frac{y}{a^2-9}$$

b) Now, find the LCD of the above.

c) Now, add: $\frac{x^2}{2a-6}+\frac{x}{a^2-a-6}+\frac{y}{a^2-9}$

11. We know we must find the LCD for adding or subtracting rational expressions. We must also find the LCD for solving rational equations. How is the use of the LCD different for solving rational equations? Why do we use it the way we do, and how does it make the problem easier to solve?

12. What are excluded values? How do we find them?

Student Activity #2 – Rational Expressions

Warm up. Simplify the following rational expressions. (Coxford & Payne, 1983)

Expression	Operation?	Factor and Other	Expand and Regroup	Complete
Example: $\frac{48}{72}\cdot\frac{28}{8}$	Multiplication	$\frac{2\cdot2\cdot2\cdot2\cdot3}{2\cdot2\cdot2\cdot3\cdot3}\cdot\frac{2\cdot2\cdot7}{2\cdot2\cdot2}$	$\frac{\cancel{2}^1}{\cancel{2}}\cdot\frac{\cancel{2}^1}{\cancel{2}}\cdot\frac{\cancel{2}^1}{\cancel{2}}\cdot\frac{\cancel{2}^1}{\cancel{2}}\cdot\frac{\cancel{2}^1}{\cancel{2}}\cdot\frac{\cancel{2}^1}{\cancel{2}}\cdot\frac{\cancel{3}^1}{\cancel{3}}\cdot\frac{7}{3}$	$\frac{7}{3}$
Example: $\frac{7y}{6z}\div\frac{9yz}{4}$	Division	$\frac{7y}{2\cdot3z}\bullet\frac{2^2}{3^2yz}$	$\frac{\cancel{2}^1}{\cancel{2}}\cdot\frac{\cancel{y}^1}{\cancel{y}}\cdot\frac{2\cdot7}{3\cdot3^2\cdot z\cdot z}$	$\frac{14}{27z^2}$
$\frac{5}{x^2}\cdot\frac{6}{10}$				
$\frac{3x^4}{4a^2y^2}\cdot\frac{8a^6y^3}{6x^7}$				
$\frac{4x^2}{3}\div\frac{x^3}{81}$				
$\frac{2x}{3y}\cdot\frac{27y^5}{16x}$				
$\frac{7m}{6n}\div\frac{9mn}{4}$				
$\frac{x^2y^5}{10z^2}\div\frac{x^4y}{6z^2}$				
$\frac{14x^2}{10y^2}\cdot\frac{15y^2}{21x^2}$				

Next, factor the following polynomials.

1. $4a^2+8a+8$

2. $4xy-4y$

3. $25x^2-36y^2$

4. $2x^2+3x+1$

5. y^3-8

6. x^2-x-12

Step up. Simplify the following rational expressions.

Expression	Operation?	Factor and Other	Expand and Regroup	Complete
Example: $\frac{a+b}{9}\cdot\frac{3}{(a+b)^2}$	Multiplication	$\frac{(a+b)}{3\cdot 3}\cdot\frac{3}{(a+b)^2}$	$\frac{\cancel{3}^1}{\cancel{3}}\,\frac{\cancel{(a+b)}^1}{3\cancel{(a+b)}(a+b)}$	$\frac{1}{3(a+b)}$
Example: $\frac{x^2-y^2}{2x+2y}\div\frac{(y-x)^2}{x^2-xy}$	Division	$\frac{(x+y)(x-y)}{2(x+y)}\cdot\frac{x(x-y)}{(y-x)^2}$	$\frac{\cancel{(x+y)}^1}{\cancel{(x+y)}}\cdot\frac{\cancel{(x-y)}^{-1}}{\cancel{(y-x)}}\cdot\frac{\cancel{(x-y)}^{-1}}{\cancel{(y-x)}}\cdot\frac{x}{2}$	$\frac{x}{2}$
$\frac{d^2-9}{d+3}\cdot\frac{d}{d-3}$				
$\frac{x^2-3x-10}{(x-2)^2}\cdot\frac{x-2}{x-5}$				
$\frac{2x^2+3x+1}{x^2-11x+24}\div\frac{2x+1}{x-3}$				
$\frac{x^2+2x+1}{x^2-4}\cdot\frac{x+2}{x^2-x-2}$				
$\frac{x^2+6xy+5y^2}{x^2+4xy+4y^2}\div\frac{x+y}{x+2y}$				
$\frac{m^2n^3-m^2n^2}{x+y}\cdot\frac{(x+y)^2}{3n-3}$				
$\frac{b^2-16}{b+2}\div\frac{b^2-8b+16}{b^3+6b^2+8b}$				
$\frac{x^2-4}{(x+2)^2}\div\frac{2-x}{3x^2+x-10}$				
$\frac{8x^3-27y^3}{2x+3y}\div\frac{4x^2+6xy+9y^2}{4x^2-9y^2}$				
$\frac{y^3-8}{y^3-4y}\cdot\frac{y^3+2y^2}{y^2+2y+4}$				
$\frac{t^2-t-6}{t^2-9}\div\frac{t^2-4}{t^2+4t+3}$				

Student Activity #3 – Rational Equations

Follow the steps in the example given:

Example: $\frac{3x}{10}+\frac{1}{4}=\frac{2x}{5}$

STEP 1: First, find the LCD: $10=2\cdot 5, 4=2^2, 5=5$. We must find the product of the largest representative of each prime. LCD= $2^2\cdot 5=20$.

STEP 2: Now, multiply both sides of the equal sign by the LCD:

$$20\left(\frac{3x}{10}+\frac{1}{4}\right)=20\left(\frac{2x}{5}\right)$$
$$20\left(\frac{3x}{10}\right)+20\left(\frac{1}{4}\right)=20\left(\frac{2x}{5}\right)$$
$$\cancel{20}^2\left(\frac{3x}{\cancel{10}}\right)+\cancel{20}^5\left(\frac{1}{\cancel{4}}\right)=\cancel{20}^4\left(\frac{2x}{\cancel{5}}\right)$$
$$6x+5=8x$$

Now it's easier to solve. This is the purpose of multiplying by the LCD.

STEP 3: Solve:

$$6x+5=8x$$
$$-6x \qquad -6x$$
$$5=2x$$
$$\frac{5}{2}=x$$

STEP 4: Check your answer by plugging it back into the original equation. You may do this step using your calculator. Remember to use grouping symbols (parentheses) where appropriate.

$$\frac{3\cdot\frac{5}{2}}{10}+\frac{1}{4}=\frac{2\cdot\frac{5}{2}}{5}$$

$$\frac{\frac{15}{2}}{10}+\frac{1}{4}=1$$

Remember, $\frac{\frac{15}{2}}{10}$ means $\frac{15}{2}\div 10$, $\frac{\cancel{15}^{3}}{2}\cdot\frac{1}{\cancel{10}^{2}}=\frac{3}{4}$. So, $\frac{3}{4}+\frac{1}{4}=1$ Check.

Now, solve the following rational equations using the same four steps from the above example. Use your calculator to check your answer.

1. $\frac{2x}{3}-\frac{1}{2}=\frac{x}{6}$

STEP 1: Find the LCD.

STEP 2: Multiply both sides by the LCD.

STEP 3: Solve.

STEP 4: Check.

2. $\frac{3x+1}{5x}=\frac{2}{3}$

STEP 1: Find the LCD.

STEP 2: Multiply both sides by the LCD.

STEP 3: Solve.

STEP 4: Check.

3. $$\frac{2}{x-2}+\frac{x-4}{x+2}=\frac{17}{x^2-4}$$

STEP 1: Find the LCD.

STEP 2: Multiply both sides by the LCD.

STEP 3: Solve.

STEP 4: Check.

4. $$\frac{2x+3}{x-1}-\frac{2x-3}{x+1}=\frac{10}{x^2-1}$$

STEP 1: Find the LCD.

STEP 2: Multiply both sides by the LCD.

STEP 3: Solve.

STEP 4: Check. What happens on your calculator? Why doesn't x = 1 work?

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