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Math 629
HW1: Chapter 9

3. Let x = the number of days the first worker can do the job alone;
Let y = the number of days the second worker can do the job alone;
Let z = the number of days the third worker can do the job alone.

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{10}$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{12}$$

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{15}$$

$$10z+10y=yz$$

$$12z+12x=xz$$

$$15y+15x=xy$$

$$yz-10z=10y; \quad z=\frac{10y}{y-10}$$

$$xz-12z=12x; \quad z=\frac{12x}{x-12}$$

$$xy-15y=15x; \quad y=\frac{15x}{x-15}$$

$$\frac{10y}{y-10}=\frac{12x}{x-12}$$

$$\frac{10\left(\frac{15x}{x-15}\right)}{\frac{15x}{x-15}-10}=\frac{12x}{x-12}; \quad \frac{\frac{150x}{x-15}}{\frac{15x-10(x-15)}{x-15}}=\frac{12x}{x-12}$$

$$\frac{150x}{\cancel{(x-15)}}\cdot\frac{\cancel{(x-15)}}{5x+150}=\frac{12x}{x-12}$$

$$\frac{30x}{x+30}=\frac{12x}{x-12}; \quad \frac{5}{x+30}=\frac{2}{x-12}$$

$$5(x-12)=2(x+30)$$

$$5x-60=2x+60$$

$$3x=120$$

$$x=40; \quad y=\frac{15\cdot 40}{40-15}=24; \quad z=\frac{12\cdot 40}{40-12}=\frac{120}{7}$$

So, the first worker can do the job alone in 40 days.
The second worker can do the job alone in 24 days.
The third worker can do the job alone in $17\frac{1}{7}$ days.

9. Divide ten into two parts such that their product is $13+\sqrt{128}$.

Let x = one part, $10 - x$ = the other part.

$$x(10-x)=13+\sqrt{128}$$
$$10x-x^2=13+\sqrt{128}$$
$$x^2=10x-(13+\sqrt{128})$$

Citing Rudolff:

$$x=\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2+c}=\frac{10}{2}\pm\sqrt{\left(\frac{10}{2}\right)^2-13-\sqrt{128}}$$
$$=5\pm\sqrt{4(3-2\sqrt{2})}=5\pm 2\sqrt{3-2\sqrt{2}}$$ These are the two parts.

14. Show that if r, s are two positive distinct roots of $x^3+d=cx$, then $t = r + s$ is a root of $x^3=cx+d$.

If r, s are two positive distinct roots of $x^3+d=cx$, then $r^3+d=cr$ and $s^3+d=cs$.
Therefore,

$$d=cr-r^3=cs-s^3$$
$$cr-cs=r^3-s^3$$
$$c(r-s)=r^3-s^3$$
$$c=\frac{r^3-s^3}{r-s}=\frac{\cancel{(r-s)}(r^2+rs+s^2)}{\cancel{(r-s)}}=r^2+rs+s^2$$

Using r + s as a root of $x^3=cx+d$,

$$(r+s)^3=(r+s)(r^2+2rs+s^2)=r^3+2r^2s+rs^2+r^2s+2rs^2+s^3$$
$$=r^3+3r^2s+3rs^2+s^3=(r^2+rs+s^2)(r+s)+d$$
$$=r^3+2r^2s+2rs^2+s^3+d$$

$$d=cr-r^3=(r^2+rs+s^2)r-r^3=r^3+r^2s+rs^2-r^3=r^2s+rs^2$$
$$So,\quad x^3=r^3+2r^2s+2rs^2+s^3+r^2s+rs^2=r^3+3r^2s+3rs^2+s^3=(r+s)^3$$

Thus, $t = r + s$ is a root of $x^3 = cx + d$.

17. Use Cardano's formula to solve $x^3 + 3x = 10$.

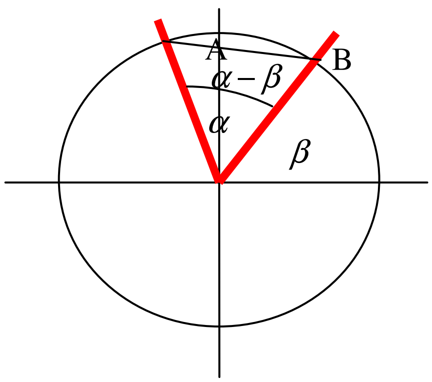
$$x = \sqrt[3]{\frac{10}{2} + \sqrt{\left(\frac{10}{2}\right)^2 - \left(\frac{-3}{3}\right)^3}} + \sqrt[3]{\frac{10}{2} - \sqrt{\left(\frac{10}{2}\right)^2 - \left(\frac{-3}{3}\right)^3}}$$

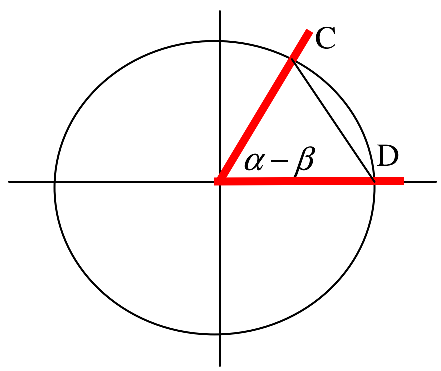
$$= \sqrt[3]{5 + \sqrt{5^2 - (-1)^3}} + \sqrt[3]{5 - \sqrt{5^2 - (-1)^3}} = \sqrt[3]{5 + \sqrt{25 + 1}} + \sqrt[3]{5 - \sqrt{25 + 1}}$$

$$= \sqrt[3]{5 + \sqrt{26}} + \sqrt[3]{5 - \sqrt{26}}$$

32. Prove that $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$.

Refer to the following diagrams, each on a unit circle. AB = CD.





$AB = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$

$= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta}$

$= \sqrt{1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} = \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$

$CD = \sqrt{[\cos(\alpha-\beta) - 1]^2 + [\sin(\alpha-\beta) - 0]^2}$

$= \sqrt{\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)} = \sqrt{1 + 1 - 2\cos(\alpha-\beta)}$

$= \sqrt{2 - 2\cos(\alpha-\beta)}$

So,

$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha-\beta)$

$-2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = -2\cos(\alpha-\beta).$

So $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. We use

$\cos(\alpha+\beta) = \cos(\alpha-(-\beta)) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

Then

$\cos(\alpha-\beta) - \cos(\alpha+\beta) = [\cos\alpha\cos\beta + \sin\alpha\sin\beta] - [\cos\alpha\cos\beta - \sin\alpha\sin\beta]$

$= 2\sin\alpha\cos\beta.$

39. Show that a projectile fired at an angle α from the horizontal follows a parabolic path.

The horizontal distance $x(t) = (v_0\cos\alpha)t$. The vertical distance is $y(t) = -at^2 + (v_0\sin\alpha)t$.

$t = \dfrac{x(t)}{v_0\cos\alpha}$, so $y(t) = y(t) = -a\left(\dfrac{x(t)}{v_0\cos\alpha}\right)^2 + v_0\sin\alpha\left(\dfrac{x(t)}{v_0\cos\alpha}\right)$

$= -a\left(\dfrac{x(t)}{v_0\cos\alpha}\right)^2 + \tan\alpha\, x(t).$ y is quadratic so the projectile follows a parabolic path.

40. d = 20000 if $\alpha = 45^0$; $d_1 = 17318 \quad if\ \alpha = 60^0\ or\ if\ \alpha = 30^0$

$$y = -a\left(\frac{x(t)}{v_0 \cos\alpha}\right)^2 + \tan\alpha x = 0.$$

$$\tan\alpha x = a\left(\frac{x}{v_0 \cos\alpha}\right)^2$$

$$\tan 45^0 \cdot 20000 = a\left(\frac{x}{v_0 \cos 45^0}\right)^2$$

$$\tan 45^0 \cdot 20000 = \frac{a \cdot 20000^2}{v_0{}^2 \cos^2 45^0}$$

$$\frac{\sin 45^0 \cdot \cos^{\cancel{2}\,1} 45^0}{\cancel{\cos 45^0}} \cdot \frac{\cancel{20000}}{20000^{\cancel{2}\,1}} = \frac{a}{v_0{}^2}$$

$$\frac{a}{v_0{}^2} = \left(\frac{\sqrt{2}}{2}\right)^2 \cdot \frac{1}{20000} = \frac{1}{40000}$$

To verify for the given angles:

$$\frac{a}{v_0{}^2} = \frac{\sin 30^0 \cos 30^0}{17318} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{17318} \approx \frac{1}{39994} \approx \frac{1}{40000}$$

$$\frac{a}{v_0{}^2} = \frac{\sin 60^0 \cos 60^0}{17318} = \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot\frac{1}{17318} \approx \frac{1}{39994} \approx \frac{1}{40000}$$