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Math 629
HW7

Georg Friedrich Bernhard Riemann

Bernhard Riemann was born on September 17, 1826 in Breselenz, Hanover, Germany, the son of a Lutheran minister, but was raised in Quickborn, Holstein, Germany. He died at age 39 on July 20, 1866 in Selasca, Italy of tuberculosis. In his short life he accomplished much as a magnificent mathematician.

Born into a poor family, Riemann was initially taught by his father. At age 14 he was admitted into a high school in Hanover, Germany. He lived with his grandmother there, but when she passed away when he was 16 he moved to go to a high school in Lüneburg, Germany. There his mathematical brilliance was noticed, and Mr. Schmalfuß, the director of the school, loaned Riemann an 859 page textbook on number theory by Legendre. He is said to have mastered the book in six days. Two years later in 1846 he went to study theology at Göttingen University. Later, with the express permission of his father, he studied mathematics in the Philosophy Department at the university. While he did hear Carl Friedrich Gauss and Moritz Stern lecture during this time, it was relatively early in his studies and they did not fully realize Riemann's brilliance yet.

He moved to Berlin University the following year to study under Dirichlet, Eisenstein, Jacobi, and Steiner. This was a fantastic opportunity for him, as this was a golden age in Germany, a great time to study mathematics under the auspices of these great mathematicians. Of these men Dirichlet influenced Riemann the most. Dirichlet style was said to be “intuitive”, lacking in extensive calculation, and Riemann followed suit.

In 1849 Riemann returned to Göttingen to work on his doctorate under Gauss. The premise of his thesis, presented in 1851, was that complex functions should be defined in such a way that, “the complex variable *w* is called a function of another complex variable *z* when its variation is such that the value of the derivative $\frac{dw}{dz}$ is independent of the value of dz.” (Katz, 2004). Further work with complex analysis led him to equations that are called the **Cauchy-Riemann equations**, in honor of both men. Cauchy derived them first but Riemann went a bit further. It goes like this:

$$\frac{dw}{dz}=\frac{du+idv}{dx+idy}=\frac{\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+i\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right)}{dx+idy}$$

$$=\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}\right)idy}{dx+idy}$$

$$\rightarrow\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\quad and\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

(Katz, 2004). Notice that this is indeed independent of *dz*.

He applied this definition to projective geometry, showing that mappings in the complex plane preserve congruent angles in the projected figure. He was able to do this by using differential geometry and showing that such a mapping implies that $\frac{du'+idv'}{dx'+idy'}=\frac{du''+idv''}{dx''+idy''}$. His proof continues to show that angles are preserved. This is done by incorporating the fact that $\frac{du'+idv'}{dx'+idy'}=\frac{\eta'}{\eta''}e^{i(\psi'-\psi'')}=\frac{du''+idv''}{dx''+idy''}=\frac{\varepsilon'}{\varepsilon''}e^{i(\phi'-\phi'')}$, where one of the critical infinitesimal distances is $\varepsilon' e^{i\phi'}$, and the other is $\eta' e^{i\psi'}$ (Katz, 2004). Of this came the **Riemann mapping theorem** which states that two simply connected regions in the complex plane can be mapped preserving angle measures.

Furthermore, the area of total curvature of any part of a surface by a closed curve is defined by using normals to a surface at a given point (perpendicular lines to the plane at any given point). Given radii of a unit circle, find the ones that are parallel to the normals, and a mapping on the surface of the sphere defines the area of the closed curvature. (Newman, 1956).

Riemann introduced and really cemented a number of ideas stemming from the use of differential geometry, including using an arbitrarily given distance between points and infinitesimals. This leads to $ds = \sum_{i,j}^{n} g_{i,j} x_i x_j$, where *ds* is the distance, *g* is the function, different for different manifolds, and this then lead to the concept of curvature tensor. Unlike in Euclidean geometry in which *through any point not on a line there exists exactly one line parallel to the given line*, in Riemannian geometry there exists no such line. Further, in this geometry (also called elliptical geometry) the sum of the angles of a triangle exceeds 180^0.

After his doctoral dissertation Riemann began working toward becoming a lecturer. In 1853 Riemann, following in the footsteps of Cauchy and Dirichlet, explored Cauchy's definition of $\int_a^b f(x)dx$. His intent was to find all functions that were integrable which had not been done to that point. He defined a necessary and sufficient condition under which a function could be integrated. Take smaller and smaller slices (widths, or *x*) of the areas beneath the curve, and if one is able to take infinitely many, then the sum will exist.

Also in 1853 he presented a paper about how the Fourier Series could be used to solve wave equations as another step toward becoming a lecturer. The question he decided to tackle was to determine which solutions to the equation $u_{xx} = c^2 u_{tt}$ could be expressed by trigonometric series. This topic had been a hot topic in the eighteenth century by such mathematical heavyweights as Euler and D'Alembert. Riemann's challenge was compounded due to a lack of foundations in analysis. This turned out to become a theme of Riemann's work, to establish general laws for both real and complex analysis. The paper was mathematically brilliant.

In 1854 he was tested on his lecturing abilities. He presented a lecture on a topic proposed by Riemann but selected by Gauss. Gauss had a choice of Riemann's three topics, namely two on the subject of electricity, and one on the subject of geometry. Normally the advisor selected the first topic presented, for which Riemann was well prepared, but instead and unexpectedly Gauss selected geometry. Gauss was very interested in how Riemann would handle the topic, given that he now believed Riemann to be a deep thinker. The lecture discussed how the laws of general geometry ought to be written as if space were any dimension, or n-dimensional. Further, geometry that had been explored up to that time could be interpreted as instances of his general geometry. The definition that he gave (now known as Riemannian Space) related what is now known as geodesics; this is the shortest distance between two points in curved space. (http://www-groups.dcs.st-and.ac.uk/~history/Riemann.html, 2007). He utilized the Pythagorean Theorem to calculate deviations of distances in his curved space world. Also, in this n-dimensional space His work in this area is said to have been indispensable to Einstein and his Theory of Relativity work. Gauss was very impressed with the lecture, discussing it at the next faculty meeting with Wilhelm Weber. While at Göttingen this second time Riemann ended up working closely with Weber in the area of theoretical physics, and Johann Listing in the area of topology, both which had a great influence on his research.

After this groundbreaking lecture, (groundbreaking in the sense that more than 60 years later Einstein made use of the concepts), Riemann was able to lecture and was given the assignment of teaching partial differential equations. He wrote home to his father that although he had only eight students initially, he was completely happy lecturing to them and that as a result of his lectures he was overcoming his shyness.

Three years after Riemann began to lecture he became a professor at Göttingen University. Soon thereafter in 1857 he published a paper on the theory of Abelian functions. He had lectured on the topic earlier to three people, one of whom was Richard Dedekind. This topic was a continuation of his work on projective and differential geometry. At this time Dirichlet was chair of the mathematics department, but when he passed away in 1859 Riemann became chair. Within a few days of this event Riemann was elected to become a member of the Berlin Academy of Sciences, and as a newly elected member he was required to submit a report on recent research. At this time he was working on the zeta function and so submitted his work on what is now called the Riemann hypothesis.

Perhaps among the most interesting and still unsolved conjectures is this Riemann hypothesis. This hypothesis began with Euler's famous formula, $\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)}$, p

are prime numbers. Riemann's zeta function $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ is a rewriting of this formula using integrals. In 1859 in a paper called, *On the number of primes less than a given quantity*, he was able to state definitively that the domain of this function is the complex plane. More work related to this function indicated that solving $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=0$ yields $s=-2n$ for n an integer provided that $n>0$. His hypothesis is that the real part of the other solutions to $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=0$ is $\frac{1}{2}$, but he was never able to prove it and it remains unproven. To date there have been more than one hundred billion zeros calculated by computer, and all of them have $\frac{1}{2}$ as their real part. This is interesting because it means that these zeros all lie on a specific line in the complex plane, and this implies that the primes are regularly distributed on the number line. (http://www.claymath.org/millennium/Riemann_Hypothesis/, 2005).

A bit of controversy surrounded the career of Riemann for awhile, and that is that he used the Dirichlet Principle in his doctoral thesis and at least one other work. The Dirichlet Principle's problem is that it did not guarantee a minimizing function, and although use of it did not negate Riemann's proofs or works, it made most mathematicians take Riemann less seriously. However, with time the controversy faded and assumed a less important role than Riemann brilliant works and their applications.

In June of 1862 Riemann married Elise Koch, and they had a daughter together. Just a few months after his marriage Riemann caught a severe cold which turned into a case of tuberculosis. From this time on, Riemann, especially during cold weather, traveled periodically to Italy. Here he would get together with some Italian mathematicians to discuss topology and analysis. However, in 1866 his health rapidly deteriorated and he died in Italy on July 20, 1866. He left behind his wife and three year old daughter.

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