Math 609
Quiz 1

1. Use Taylor's theorem with $n = 3$ and $x_0 = 1$ to estimate $\ln x$ for $x = 0.75$. What is $P_3(x)$ and $P_3(0.75)$? Estimate the maximum error in this approximation. Do not evaluate $\ln x$ for any value of x other than $x = 1$.

Solution: Taylor's theorem states the following: Suppose $f \in C^n[a,b]$, that f^{n+1}exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x - x_0)^k \text{ and}$$

$$R_n(x) = \frac{f^{(n+1)}\xi(x)}{(n+1)!}(x - x_0)^{n+1}$$

(Burden, p.10)

We know that $f(x) = \ln x$ is continuous on [a,b], b > a > 0. Following we show that f^{n+1}=f''''exists on $[a,b]$. Set $x_0 \in [a,b]$.

To find $P_3(x)$ and $P_3(0.75)$, we find the following:

$$f(x) = \ln x;\ f'(x) = \frac{1}{x};\ f''(x) = -\frac{1}{x^2};\ f'''(x) = \frac{2}{x^3} = \frac{2!}{x^3};\ f''''(x) = -\frac{6}{x^4} = -\frac{3!}{x^4}$$

Now,

$$P_3(x) = \sum_{k=0}^{3} \frac{f^k(x_0)}{k!}(x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$

$$= \ln(x_0) + \frac{1}{x_0}(x - x_0) + \frac{-1}{2!(x_0)^2}(x - x_0)^2 + \frac{2!}{3!(x_0)^3}(x - x_0)^3$$

$$= \ln(x_0) + \frac{1}{x_0}(x - x_0) - \frac{1}{2(x_0)^2}(x - x_0)^2 + \frac{1}{3(x_0)^3}(x - x_0)^3$$

$$= \ln(x_0) + \frac{(x - x_0)}{x_0} - \frac{(x - x_0)^2}{2(x_0)^2} + \frac{(x - x_0)^3}{3(x_0)^3}$$

$$= \ln 1 + \frac{(x-1)}{1} - \frac{(x-1)^2}{2(1)^2} + \frac{(x-1)^3}{3(1)^3} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

And

$$P_3(0.75) = f(1) + f'(1)(0.75-1) + \frac{f''(1)}{2!}(0.75-1)^2 + \frac{f'''(1)}{3!}(0.75-1)^3$$

$$= \ln(1) + \frac{(0.75-1)}{1} - \frac{(0.75-1)^2}{2(1)^2} + \frac{(0.75-1)^3}{3(1)^3}$$

$$= 0 - 0.25 - \frac{(-0.25)^2}{2} + \frac{(-0.25)^3}{3} = -.2864583333$$

This represents our Taylor polynomial, an estimate of $\ln x$ for $x = 0.75$.

To find the estimated maximum error we must first calculate the truncation error, $R_n(x)$.

$$R_n(x) = \frac{f^{(n+1)}\xi(x)}{(n+1)!}(x-x_0)^{n+1} \text{ so}$$

$$R_3(x) = \frac{f^{(4)}\xi(x)}{4!}(x-x_0)^4 = -\frac{3!}{4!\xi(x)^4}(x-x_0)^4 = -\frac{1}{4\xi(x)^4}(x-x_0)^4$$

So for our function, the estimated maximum error is found when $\xi(x)$ is smallest :

$$R_3(0.75) = \frac{f^{(4)}\xi(0.75)}{4!}(0.75-1)^4 = -\frac{3!}{4!1^4}(0.75-1)^4 = -\frac{1}{4}(0.75-1)^4$$

$$= -\frac{1}{4}(-0.25)^4 = -0.0009765625000$$

Since $R_3(x) = -\frac{1}{4x^4}(x-x_0)^4$ and the fourth derivative is $f''''(x) = -\frac{6}{x^4} = -\frac{3!}{x^4}$, and since the number $\xi(x)$ is between x_0 and x, the error is bounded by $R_3(0.75) = -0.0009765625000$ and $R_3(1) = 0$. Therefore, the maximum error is

$$|R_3(0.75)| = |-0.0009765625000| = 0.0009765625000 = 9.765625000 \text{ X } 10^{-4}$$

Moreover, the Taylor polynomial for n = 3 with remainder term about $x_0 = 1$ is

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{1}{4\xi(x)^4}(x-1)^4 \text{ for some } \xi(x) \text{ between 1 and } x.$$

2. Use Taylor's theorem with $n = 3$ and $x_0 = 1$ to estimate $\ln x$ for $x = 1.5$. What is $P_3(x)$ and $P_3(1.5)$? Estimate the maximum error in this approximation. Do not evaluate $\ln x$ for any value of x other than x = 1.

Solution: Taylor's theorem states the following: Suppose $f \in C^n[a,b]$, that f^{n+1}exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k \text{ and}$$

$$R_n(x) = \frac{f^{(n+1)}\xi(x)}{(n+1)!}(x-x_0)^{n+1}$$

(Burden, p.10)

We know that $f(x) = \ln x$ is continuous on [a,b]. Following we show that f^{n+1}=f''''exists on $[a,b]$. Set $x_0 \in [a,b]$.

To find $P_3(x)$ and $P_3(1.5)$, we find the following:

$$f(x) = \ln x;\ f'(x) = \frac{1}{x};\ f''(x) = -\frac{1}{x^2};\ f'''(x) = \frac{2}{x^3} = \frac{2!}{x^3};\ f''''(x) = -\frac{6}{x^4} = -\frac{3!}{x^4}$$

Now,

$$P_3(x) = \sum_{k=0}^{3} \frac{f^k(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$= \ln(x_0) + \frac{1}{x_0}(x-x_0) + \frac{-1}{2!(x_0)^2}(x-x_0)^2 + \frac{2!}{3!(x_0)^3}(x-x_0)^3$$

$$= \ln(x_0) + \frac{1}{x_0}(x-x_0) - \frac{1}{2(x_0)^2}(x-x_0)^2 + \frac{1}{3(x_0)^3}(x-x_0)^3$$

$$= \ln(x_0) + \frac{(x-x_0)}{x_0} - \frac{(x-x_0)^2}{2(x_0)^2} + \frac{(x-x_0)^3}{3(x_0)^3}$$

$$= \ln 1 + \frac{(x-1)}{1} - \frac{(x-1)^2}{2(1)^2} + \frac{(x-1)^3}{3(1)^3} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

And

$$P_3(1.5) = f(1) + f'(1)(1.5-1) + \frac{f''(1)}{2!}(1.5-1)^2 + \frac{f'''(1)}{3!}(1.5-1)^3$$

$$= \ln(1) + \frac{(1.5-1)}{1} - \frac{(1.5-1)^2}{2(1)^2} + \frac{(1.5-1)^3}{3(1)^3}$$

$$= 0 + .5 - \frac{(.5)^2}{2} + \frac{(.5)^3}{3} = 0.4166666667$$

To find the maximum error we must first calculate $R_n(x)$.

$$R_n(x) = \frac{f^{(n+1)}\xi(x)}{(n+1)!}(x-x_0)^{n+1} \text{ so}$$

$$R_3(x) = \frac{f^{(4)}\xi(x)}{4!}(x-x_0)^4 = -\frac{3!}{4!\xi(x)^4}(x-x_0)^4 = -\frac{1}{4\xi(x)^4}(x-x_0)^4$$

So for our function the estimated maximum error is found when $\xi(x)$ is smallest :

$$R_3(1.5) = \frac{f^{(4)}\xi(1.5)}{4!}(1.5-1)^4 = -\frac{3!}{4!1^4}(1.5-1)^4 = -\frac{1}{4}(1.5-1)^4$$

$$= -\frac{1}{4}(.5)^4 = -0.01562500000$$

Since $R_3(x) = -\frac{1}{4x^4}(x-x_0)^4$ and the fourth derivative is $f''''(x) = -\frac{6}{x^4} = -\frac{3!}{x^4}$, and since the number $\xi(x)$ is between x_0 and x, the error is bounded by $R_3(1.5) = -0.01562500000$ and $R_3(1) = 0$. Therefore, the maximum error is $|R_3(1.5)| = |-0.01562500000| = 0.01562500000$ =1.5625000 X 10^{-2}

3. Use Taylor's theorem with $x_0 = 1$ to estimate $\ln 1.25$ with the maximum estimated error less than 10^{-4}. How large must the degree of the Taylor polynomial be to ensure this accuracy? What is this Taylor polynomial? What is its value at 1.25? Do not evaluate $\ln x$ for any value of x other than x = 1.

Solution: To find the maximum estimated error less than 10^{-4} we will analyze the truncation error.

Since $R_3(x) = -\frac{1}{4x^4}(x-x_0)^4$ and the fourth derivative is $f''''(x) = -\frac{6}{x^4} = -\frac{3!}{x^4}$, and since the number $\xi(x)$ is between x_0 and x, the error is bounded by

$$R_3(1.25) = \frac{f^{(4)}\xi(1.25)}{4!}(1.25-1)^4 = -\frac{3!}{4!1^4}(1.25-1)^4 = -\frac{1}{4}(1.25-1)^4$$

$$= -\frac{1}{4}(.25)^4 = -0.0009765625000 = -9.765625000 \text{ X } 10^{-4}$$

$R_3(1.25) = -0.01562500000$ and $R_3(1) = 0$. Therefore, the maximum error for the the third degree polynomial estimate is

$\left|R_3(1.25)\right| = \left|-0.0009765625000\right| = 0.0009765625000 = 9.765625000 \text{ X } 10^{-4} > 10^{-4}$.

This is too large and so we must look at the next Taylor polynomial estimate, $P_4(x)$. Its associated truncation error,

$R_4(x) = \frac{f^{(5)}\xi(x)}{5!}(x-x_0)^5 = -\frac{4!}{5!x^5}(x-x_0)^5 = -\frac{1}{5x^4}(x-x_0)^5$, so

$$R_4(1.25) = \frac{f^{(5)}\xi(1.25)}{5!}(1.25-1)^5 = \frac{4!}{5!1^5}(1.25-1)^5 = \frac{1}{5}(1.25-1)^5$$

$$= \frac{1}{5}(.25)^5 = 0.0001953125000 = 1.953125000 \text{ X } 10^{-4}$$

Still too large.

$$R_5(1.25) = \frac{f^{(6)}\xi(1.25)}{6!}(1.25-1)^6 = -\frac{5!}{6!1^6}(1.25-1)^6 = -\frac{1}{6}(1.25-1)^6$$

$$= -\frac{1}{6}(.25)^6 = -0.00004069010417 = -4.069010417 \text{ X } 10^{-5}$$

$$\left|R_5(1.25)\right| = \left|-0.00004069010417\right| = 0.00004069010417 = 4.069010417 \text{ X } 10^{-5} < 10^{-4}$$

Since this meets the criterion of having a maximum error less than 10^{-4}, we conclude that the Taylor polynomial is of degree 5 and is given by the following:

$$P_5(x) = \ln(x_0) + \frac{(x-x_0)}{x_0} - \frac{(x-x_0)^2}{2(x_0)^2} + \frac{(x-x_0)^3}{3(x_0)^3} - \frac{(x-x_0)^4}{4(x_0)^4} + \frac{(x-x_0)^5}{5(x_0)^5}$$

$$= \ln(1) + \frac{(x-1)}{1} - \frac{(x-1)^2}{2(1)^2} + \frac{(x-1)^3}{3(1)^3} - \frac{(x-1)^4}{4(1)^4} + \frac{(x-1)^5}{5(1)^5}$$

$$= \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}$$

Its value at 1.25 is given by:

$$P_5(1.25) = f(1) + f'(1)(1.25-1) + \frac{f''(1)}{2!}(1.25-1)^2 + \frac{f'''(1)}{3!}(1.25-1)^3 + \frac{f''''(1)}{4!}(1.25-1)^4$$
$$+\frac{f'''''(1)}{5!}(1.25-1)^5$$
$$= \ln(1) + \frac{(1.25-1)}{1} - \frac{(1.25-1)^2}{2(1)^2} + \frac{(1.25-1)^3}{3(1)^3} - \frac{(1.25-1)^4}{4(1)^4} + \frac{(1.25-1)^5}{3(1)^5}$$
$$= 0 + .25 - \frac{(.25)^2}{2} + \frac{(.25)^3}{3} - \frac{(.25)^4}{4} + \frac{(.25)^5}{5} = 0.2231770833$$

4. The function $f(x)$ and all of its derivatives are continuous on [0, 10]. You know that $f(0)=0,\ f(2)=0,\ f(3)=0,\ f(6)=0,$ and $f(8)=0$. At how many points must the first derivative of *f(x)* be zero? At how many points must the second derivative of *f(x)* be zero? At how many points must the third derivative of *f(x)* be zero, and so on? Justify your answers.

Solution: Since $f(x)$ and all of its derivatives are continuous on [0, 10], and since $f(0)=f(2)=0$, we know a number a_1 in (0, 2) exists with $f'(a_1)=0$ (Rolle's Theorem).
Similarly, since $f(2)=f(3)=0$, a number a_2 in (2, 3) exists with $f'(a_2)=0$.
Since $f(3)=f(6)=0$, a number a_3 in (3, 6) exists with $f'(a_3)=0$.
Since $f(6)=f(8)=0$, a number a_4 in (6, 8) exists with $f'(a_4)=0$. Therefore, the first derivative of *f(x)* must be zero at at least 4 points.

As for the second derivative of *f(x),* since the first derivative is continuous on the interval [0, 10], there must be at least 3 points at which the second derivative of *f(x)* must be zero. Indeed, extending Rolle's theorem (Generalized Rolle's Theorem), since $f'(a_1)=f'(a_2)=0$ there exists a number b_1 in (a_1,a_2) with $f''(b_1)=0$.
Since $f'(a_2)=f'(a_3)=0$ there exists a number b_2 in (a_2,a_3) with $f''(b_2)=0$.
Since $f'(a_3)=f'(a_4)=0$ there exists a number b_3 in (a_3,a_4) with $f''(b_3)=0$.

As for the third derivative of *f(x),* since the second derivative is continuous on the interval [0, 10], there must be at least 2 points at which the second derivative of *f(x)* must be zero. Indeed, using the Generalized Rolle's Theorem, since $f''(b_1)=f''(b_2)=0$ there exists a number c_1 in (b_1,b_2) with $f'''(c_1)=0$.
Since $f''(b_2)=f''(b_3)=0$ there exists a number c_2 in (b_2,b_3) with $f'''(c_2)=0$.

By the Generalized Rolle's Theorem, since $f'(x)$ is zero at the 4 distinct points in [0, 10], then a number *d* exists in [0, 10] with $f^4(d)=0$. So we know there is at least one point at which the fourth derivative of *f(x)* must be zero.