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Math 646
Book Report

Solving Mathematical Problems

by Terence Tao

Solving Mathematical Problems, by Terence Tao, is an updated version of a problem-solving book at the Mathematics Olympiad level, written originally when Dr. Tao was just 15 years old. Updates include some corrections and more examples. The book is small, approximately 100 pages, and divided into six chapters. It contains detailed solutions, thoughts that went into the process of solving, and 29 unsolved challenge problems.

Dr. Tao is not only a superb mathematician, having recently won the Fields Medal in 2006, but he is also a great teacher. His book is easy to read and follow, and his suggested problem solving techniques are clear and step-by-step. Although his genius is evident, this book takes a friendly approach, and allows the reader to see many different paths to solving a particular problem. In viewing all the paths, one can make some conclusions, or at least come up with some ideas, about what type of approach works best with specific types of problems.

The book begins by clearly stating that it is important to understand the problem. There are three types of problems, according to Tao (p. 1):

- 'Show that….' or 'Evaluate…' problem
- 'Find a …' or 'Find all….' Requiring someone to satisfy a condition.
- 'Is there a …' questions, requiring a proof or a counterexample.

These three are listed in order of difficulty, from easiest to hardest. Next, Tao details important things the student must do to solve problems. These are: understand the data, select good notations, write down what you know, and draw a diagram.

There are reasons for the thing Tao suggests; a favorites is that, “the physical act of writing down of what you know can trigger new inspirations and connections.” (p. 3) Sometimes, with more difficult problems, Tao suggests that altering the problem somewhat may help clarify and ultimately lead to the solution.

Specific types of problems covered include number theory, algebra and analysis, Euclidean Geometry, Analytic Geometry, and “Sundry” examples. Under “number theory”, Tao covers sums of digit problems, powers of numbers and their digits, Diophantine equations, and sums of powers. The difference between algebraic problem solving and number theory is that number theory seems to get results out of nowhere. Algebra manipulates equations.

The keys to solving number theory problems involve modular arithmetic, integer division, and integral factorization. Use of division rules seem to be used extensively. Number Theory problems usually require break down to either reduce the number of possible solutions or to more easily analyze the problem. Sometimes, an initial idea does not make the problem easier, and must be abandoned. Tao implies that one must try things, but be ready try an alternate course. Guess easier answers first. “If right, you saved a lot of time… If wrong, you were doomed to a long haul anyway.” (p. 14)

Through analysis and detailing his thought process on several problems, one can gain insight into preferred paths to take for specific types of problems. For example, Tao walks through the process of solving a particular problem that involved both the individual digits of the powers of two, and the rearrangement of digits. While the rearrangements of the digits seems the like a good path, quickly Tao shows us that using modular arithmetic is more useful and simplifies the problem by radically reducing the number of possible solutions. The example, on p.14, is the following problem (Taylor 1989, p.37):

Is there a power of 2 such that its digits could be rearranged and made into another power of 2? (No zeros are allowed in the leading digit: for example, 0032 is not allowed.)

Tao discusses his thought process: the fact that there are too many possibilities in digit rearrangement, and the digits of the powers of two are hard to determine. So, he narrows down the problem by using divisibility rules. First he clarifies that there are really only two things we care about (i.e.: powers of two and digit switching), and then decides that the powers of two are easier to deal with. Then he says that digit switching restricts the number of digits, and although this is a rather obvious fact, he encourages the problem-solver to write it down, because it may not come to mind later. He changes the question to one meant to make the original problem easier. He asks, "Is there a power of 2 such that there is another power of 2 with the same number of digits as the first power of 2?" With this general question he begins discussing all the powers of two with x number of digits. There are not that many within the smaller powers. We need to notice that the digits, as they won't change with digit rearrangement, have the same digit sum. Tao makes a table comparing the powers to their respective digit sum, and next a table with the power mod 9; all the data is here:

Power of 2	**Digit-sum**	**(mod 9) Remainder**	**Power of 2**	**Digit-sum**	**(mod 9) Remainder**	**Power of 2**	**Digit-sum**	**(mod 9) Remainder**
1	1	**1**	256	13	**4**	65536	25	**7**
2	2	**2**	512	8	**8**	131072	14	**5**
4	4	**4**	1024	7	**7**	262144	19	**1**
8	8	**8**	2048	14	**5**	524288	29	**2**
16	7	**7**	4096	19	**1**	1048576	31	**4**
32	5	**5**	8192	20	**2**			
64	10	**1**	16384	22	**4**			
128	11	**2**	32768	26	**8**			

By using mod 9 a pattern emerges. It becomes obvious that what we need to prove is that "no two powers of 2 have the same remainder (mod 9) and the same number of digits." (p. 18). There are several in our table with the same remainder, but they are spread so far apart that they don't have the same number of digits. He explains that using modular arithmetic:

$2^{n+6} = 2^n 2^6 = 2^n \times 64 = 2^n$ (mod 9) because 64 = 1 (mod 9). He concludes with the proof (p. 18):

> *PROOF: Suppose two powers of 2 are related by digit-switching. This means that they have the same number of digits, and also have the same digit-sum (mod9). But the digit-sums (mod 9) are periodic with a period of 6, with no repetitions with in any given period, so the two powers are at least six steps apart. But then it is impossible for them to have the same number of digits, a contradiction.*

Tao also covers Diophantine equations, suggesting that using modular arithmetic and factorization are key. When working on problems that have a parameter, one should keep in mind the idea of periodicity; if one can find a repeating modulo pattern, one needs only check those parts that repeat.

When Tao gets into the next section dealing with function analysis, he explains that since these types of questions generally deal with algebra, one usually tries to manipulate them as one would algebra. But, with functions, he generally begins by finding terms or analyzing the first few cases, then tries to figure out if there is something else he needs to prove to end up solving the problem. In these types of problems Tao often uses proof by induction. The other topic covered in this chapter is problem solving involving polynomials; here Tao suggests a trial and error approach, and manipulation of equations.

In the Euclidean Geometry chapter Tao says that working on the angles is generally easier than working on side lengths. However, he says there are two approaches to these types of problems: direct approach, whereby one hammers out the solution by working on the given pieces until it looks like what you are trying to show, and the "change the objective" approach, which he implies works well for obscure problems. One can work forward, figure out sides and angles systematically, or backward, by changing the objective into a simpler objective that is easier to work with, and work backwards.

In the Analytic Geometry chapter, Tao says that the problems presented there involve geometric shapes but require other branches of mathematics to solve, such as algebra, induction, and inequalities. He distinguishes the categories of Euclidean vs. Analytic by saying essentially that if you are not asked to find out something about lengths or degree measures, but something else such as squares of lengths, then you are in the realm of Analytic Geometry. This separation is very helpful and provides an idea of how to start. Here he draws diagrams and then comes up with several possible approaches to the problem at hand. He also states truths about the diagrams and tries to find a pathway to the solution.

In the last chapter, Tao covers “sundry examples” which he says are “not quite game theory, not quite combinatorics, not quite linear programming” (p. 83). He suggests that every piece of information, even if useless-looking, ought to be written down because many of the sundry problems need to be worked as equations. Any truth that can help, such as “numbers of a type of animal cannot be negative”, may prove vital later, or at least narrow the problem thus simplifying it.

After each of the types of problems are discussed and solved, Tao also submits several more of a similar type without solution given, presumably to challenge the reader. What is enjoyable about the book is that the author allows the reader to experience his thought process, wrong turns included. It is very helpful to recognize that sometimes one must travel down a path for a bit to see whether it is will be useful or not. Overall, Tao’s book is recommended reading for anyone wishing to improve problem-solving skills.